Chapter 7

Wavelets and Multiresolution Processing

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Some slides and illustrations from Dr. Jimin Liang and Dr. Nawapak Eua-Anant
Wavelets and Multiresolution Processing

Preview

- Good old Fourier transform
  - A transform where the basis functions are *sinusoids*, hence localized in frequency but not localized in time; in transform domain the temporal information is lost
- Short-time Fourier transform
- Wavelets
  - *Varying frequency* and *limited duration*; an attempt to reveal both the frequency and temporal information in the transform domain
- Multiresolution analysis
  - Signal (image) representation at *multiple resolutions*
Background

Images

- Connected regions of similar texture combined to form different objects
  - Small sized or low in contrast objects: examined at high resolution
  - Large sized or high in contrast: examined at a coarse view
  - Several resolutions needed to distinguish between different objects

- Mathematically images are 2-D arrays of intensity values of locally varying statistics (resulting from edges, homogenous regions, etc. — see Fig. 7.1)
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Histograms computed in local regions

FIGURE 7.1 A natural image and its local histogram variations.

Basic ideas of linear transformation

- To change the coordinate system in which a signal (image/video) is represented in order to make it much better suited for processing (e.g. compression).
- To represent all the useful signal features and important phenomena in as compact manner as possible.
- Important to compact the bulk of the signal energy into the fewest number of transform coefficients.
Four properties of a Good Transform:

- Decorrelate the image pixels
- Provide good energy compaction
- Desirable to be orthogonal
- Efficient computational algorithm should exist

There is no universal good transform. Among block transforms – KLT is statistically optimal (first 3 properties fulfilled, but 4th – no). Instead, in image processing, fixed "suboptimal" transforms are used, such as DFT, DCT, lapped transforms, wavelets, etc.
Classical Fourier transform:

- ‘uncertainty principle’ $u[k]$ has a narrow support (localized), then $U(.)$ has a wide support (non-localized), and vice versa.
- Notion of ‘frequency that varies with time’ not accommodated.

$$U(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} u[k].e^{-j\omega k}, \quad 0 \leq \omega \leq 2\pi$$

$$u[k] = \frac{1}{2\pi} \int_{0}^{2\pi} U(e^{j\omega}).e^{j\omega k} d\omega$$
Fourier transform acts as an optical prism

Fourier transform (Fourier spectrum)

Signal → Fourier Transform → Constituent sinusoids of different frequencies
**Short-time Fourier (STFT) transform**

\[
U_{STFT}(e^{j\omega}, n) = \sum_{k=-\infty}^{+\infty} u[k] . w[k - n] . e^{-j\omega k} \quad 0 \leq \omega \leq 2\pi, \quad -\infty < n < +\infty
\]

\(w[k]\) is a window function (Gabor transform \(w[k]\) is Gaussian)
Short-time Fourier (STFT) transform

STFT can be re-written as

\[ U_{STFT}(e^{j\omega}, n) = e^{-j\omega n} \sum_{k=-\infty}^{+\infty} u[k] \cdot w[k - n] \cdot e^{-j\omega(k-n)} \]

This is a convolution with a filter \[ w[-n] \cdot e^{j\omega n} \]

Computing it for discrete frequencies \[ \omega_k = k \cdot \frac{2\pi}{N} \], \[ k = 0, \ldots, N - 1 \]

make a set of filters \[ h_k[n] = h_0[n] \cdot e^{j2\pi k \cdot n / N} \], \[ h_0[n] = w[-n] \]

This leads to a DFT-modulated filter bank with a prototype filter – window function
Image Pyramids

- Creation by iterated approximations and interpolations (predictions)
- The approximation output is taken as the input for the next resolution level
- For a number of levels $P$, two pyramids: approximation (Gaussian) and prediction residual (Laplacian), are created
- approximation filters could be:
  - averaging
  - low-pass (Gaussian)

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Gaussian and Laplacian Pyramids

Starting image is $512 \times 512$; convolution kernel (approximation kernel) is $5 \times 5$ Gaussian. Different resolutions are appropriate for different image objects (window, vase, flower, etc.)

FIGURE 7.3 Two image pyramids and their statistics: (a) a Gaussian (approximation) pyramid and (b) a Laplacian (prediction residual) pyramid.

Subband Coding

- The image is decomposed into a set of band-limited components (subbands).
- Two-channel, perfect reconstruction system; two sets of half-band filters
- **Analysis:** $h_0$ – low-pass and $h_1$ – high-pass and **Synthesis:** $g_0$ – low-pass and $g_1$ – high-pass
Brief reminder of Z-transform properties

- The Z-Transform of a sequence $x(n)$, $n = 0, 1, 2, \ldots$ is
  \[ X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \]

- Z-Transform of sequences **downsampled** or **upsampled** by a factor of 2

  \[ x_{\text{down}}(n) = x(2n) \Longleftrightarrow X_{\text{down}}(z) = \frac{1}{2} \left[ X(z^{1/2}) + X(z^{-1/2}) \right] \]

  \[ x_{\text{up}}(n) = \begin{cases} x(n/2) & n = 0, 2, 4, \ldots \\ 0 & \text{otherwise} \end{cases} \Longleftrightarrow X_{\text{up}}(z) = X(z^2) \]
• The system output can be expressed as

\[
\hat{X}(z) = \frac{1}{2} G_0(z) \left[ H_0(z) X(z) + H_0(-z) X(-z) \right] \\
+ \frac{1}{2} G_1(z) \left[ H_1(z) X(z) + H_1(-z) X(-z) \right] \\
= \frac{1}{2} \left[ H_0(z) G_0(z) + H_1(z) G_1(z) \right] X(z) \\
+ \frac{1}{2} \left[ H_0(-z) G_0(z) + H_1(-z) G_1(z) \right] X(-z)
\]

aliasing term
In order for the system to yield $\hat{x}(n) = x(n)$ or, equivalently, $\hat{X}(z) = X(z)$ it has to be

\[
H_0(z)G_0(z) + H_1(z)G_1(z) = 2
\]
\[
H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0
\]
Define the *analysis modulation matrix*

\[ \mathbf{H}_m(z) = \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \]

Constraints then become

\[ \begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \mathbf{H}_m(z) = \begin{bmatrix} 2 & 0 \end{bmatrix} \]

If \( \mathbf{H}_m(z) \) is nonsingular

\[ \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2}{\det(\mathbf{H}_m(z))} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix} \]
Perfect Reconstruction Filter Families

QMF – quadrature-mirror filters
CQF – conjugate-quadrature filters

<table>
<thead>
<tr>
<th>Filter</th>
<th>QMF</th>
<th>CQF</th>
<th>Orthonormal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_0(z)$</td>
<td>$H_0^2(z) - H_0^2(-z) = 2$</td>
<td>$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2$</td>
<td>$G_0(z^{-1})$</td>
</tr>
<tr>
<td>$H_1(z)$</td>
<td>$H_0(-z)$</td>
<td>$z^{-1}H_0(-z^{-1})$</td>
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</tr>
<tr>
<td>$G_1(z)$</td>
<td>$-H_0(-z)$</td>
<td>$zH_0(-z)$</td>
<td>$-z^{-2^k+1}G_0(-z^{-1})$</td>
</tr>
</tbody>
</table>

**TABLE 7.1**
Perfect reconstruction filter families.
2-channel filter bank: Analysis bank

- $H'$ is the **lowpass** filter and $G'$ is the **highpass** filter.
- $\downarrow 2$ is the **downsampling** operator: $(1 \ 3 \ 4 \ 6 \ 5) \rightarrow (1 \ 4 \ 5)$.

**Diagram:**

- **Lowpass channel**:
  - Input $x$ goes through $H'$.
  - Downsampling $\downarrow 2$ produces $s$.

- **Highpass channel**:
  - Input $x$ goes through $G'$.
  - Downsampling $\downarrow 2$ produces $d$. 
- $H$ is the lowpass filter and $G$ is the highpass filter.
- $\uparrow 2$ is the upsampling operator: $(1\ 4\ 5) \rightarrow (1\ 0\ 4\ 0\ 5)$.

Diagram:

- **Lowpass channel**
  - $s$ $\rightarrow$ $\uparrow 2$ $\rightarrow$ $H$

- **Highpass channel**
  - $d$ $\rightarrow$ $\uparrow 2$ $\rightarrow$ $G$

Output: $y$
A biorthogonal filter bank

Biorthogonal (or perfect) filter bank: if $y = x$ for all inputs $x$. 
An orthogonal filter bank

Orthogonal filter bank: if it is biorthogonal, and both analysis filters $H'$ and $G'$ are the time reversals of the synthesis filters $H$ & $G$: $H=(1, 2, 3) \rightarrow H'=(3, 2, 1)$. 
Separable filtering in 2-D (images)

- Vertical followed by horizontal filtering
- Four output subbands:
  - approximation, and vertical, horizontal, and diagonal detail subbands

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Example: four band subband filtering of an image

Aliasing is presented in the vertical and horizontal detail subbands. It is due to the down-sampling and will be canceled during the reconstruction stage.

FIGURE 7.7 A four-band split of the vase in Fig. 7.1 using the subband coding system of Fig. 7.5.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Example: 8-tap orthonormal filter designed by Daubechies

FIGURE 7.6 The impulse responses of four 8-tap Daubechies orthonormal filters.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelet transform decomposes a signal into a set of basis functions.

These basis functions are called wavelets.

Transform a continuous function into a highly redundant function of two variables – translation ($t$) and scale ($s$).

$$W_\psi(s, \tau) = \int_{-\infty}^{\infty} f(x) \psi_{s,\tau}(x)$$

where $$\psi_{s,\tau}(x) = \frac{1}{\sqrt{s}} \psi\left(\frac{x - \tau}{s}\right)$$

$$f(x) = \frac{1}{C_\psi} \int_{0}^{\infty} \int_{-\infty}^{\infty} W_\psi(s, \tau) \frac{\psi_{s,\tau}(x)}{s^2} \, d\tau \, ds$$

where $$C_\psi = \int_{-\infty}^{\infty} \frac{|\Psi(\omega)|^2}{|\omega|} \, d\omega$$
Comparison between Fourier Transform and Continuous WT

**FIGURE 7.14** The continuous wavelet transform (c and d) and Fourier spectrum (b) of a continuous one-dimensional function (a).

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
The standard Fourier Transform (FT) decomposes the signal into individual frequency components.

- The Fourier basis functions are infinite in extent.
- FT can never tell when or where a frequency occurs.
- Any abrupt changes in time in the input signal $f(t)$ are spread out over the whole frequency axis in the transform output $F(\omega)$ and vice versa.

WT uses short window at high frequencies and long window at low frequencies (recall $a$ and $b$ in previous formula). It can localize abrupt changes in both time and frequency domains.
Transform pair (forward and inverse transform) applicable for discrete signals (or sampled signals)

\[
W_\phi(j_0, k) = \frac{1}{\sqrt{M}} \sum_x f(x) \phi_{j_0, k}(x)
\]

\[
W_\psi(j, k) = \frac{1}{\sqrt{M}} \sum_x f(x) \psi_{j, k}(x)
\]

\[
f(x) = \frac{1}{\sqrt{M}} \sum_k W_\phi(j_0, k) \phi_{j_0, k}(x) + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi(j, k) \psi_{j, k}(x)
\]

Here \( x \) is a discrete variable: \( x=0,1,2,\ldots M-1 \) Normally, \( M=2^J; j_0=0; j=0,1,2,\ldots J-1 \) and \( k=0,1,2,\ldots 2^j-1 \) \( W_\phi \) and \( W_\psi \) are referred as approximation and detail coefficients respectively.
The Haar “aperture” function is

\[ \psi_{\text{harr}}(x) = 1_{0 \leq x < 1/2}(x) - 1_{1/2 \leq x < 1}(x). \]

Haar’s theorem (1905):

- All Haar wavelets \( \psi_{j,k}^{\text{harr}} \), together with the constant function \( 1 \), consist into an orthonormal basis for the Hilbert space of all square integrable functions on \([0, 1]\).
Haar Transform (cont’d)

- Can be expressed in matrix form as
  \[ T = HFH \]
  where \( F \) is an \( N \times N \) image, \( H \) is an \( N \times N \) transformation matrix, and \( T \) is the resulting \( N \times N \) transform

- Matrix \( H \) contains the Haar basis functions \( h_k(z) \) defined over the closed interval \( z \in [0,1] \) and for \( k = 0,1,2,\ldots,N-1, \ N = 2^n \)
Example: $2 \times 2$ and $4 \times 4$ Transform Matrix

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{H}_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & \sqrt{2} \end{bmatrix}$$
Haar Functions in a Discrete Wavelet Transform

FIGURE 7.8  (a) A discrete wavelet transform using Haar basis functions. Its local histogram variations are also shown;  
(b)–(d) Several different approximations (64 × 64, 128 × 128, and 256 × 256) that can be obtained from (a).

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelets work for decomposing signals (such as images) into hierarchy of increasing resolutions: as we consider more layers, we get more and more detailed look at the image.
Another Example: The Shannon wavelets

- The Shannon’s “aperture” function is:
  \[ \psi^{\text{Shannon}}(x) = 2 \text{sinc}(2x) - \text{sinc}(x). \]

- **Theorem:**
  \[ \{ \psi^{\text{Shannon}}_{j,k}(x) : j, k \in \mathbb{Z} \} \text{ is an orthonormal basis of } L_2(\mathbb{R}). \]
How to visualize the orthonormal basis?

**Answer:** go to the Fourier domain!

\[ \psi_{\text{shannon}}(x) = 2 \text{sinc}(2x) - \text{sinc}(x). \]

According to Shannon:

- All signals bandlimited to \((-\pi, \pi)\) can be represented by \(\text{sinc}(x-n)\)...
- those bandlimited to \((-2\pi, \pi) \cup (\pi, 2\pi)\), by \(\psi(x-n)\).
- those bandlimited to \((-4\pi, 2\pi) \cup (2\pi, 4\pi)\), by \(\psi_{1,n} = \sqrt{2}\psi(2x-n)\).
According to Shannon:

- All signals bandlimited to \((-\pi, \pi)\) can be represented by \(\text{sinc}(x-n)\)...
- those bandlimited to \((-2\pi, -\pi) \cup (\pi, 2\pi)\), by \(\psi(x-n)\).
- those bandlimited to \((-4\pi, -2\pi) \cup (2\pi, 4\pi)\), by \(\psi_{1,n} = \sqrt{2}\psi(2x-n)\).

### Shannon wavelets (cont’d)

- represented by \(\psi_{1,n} = \sqrt{2}\psi(2x-n)'s\).
- represented by \(\psi(x-n)'s\).
- represented by \(\psi_{-1,n} = 1 / \sqrt{2}\psi(x/2 - n)'s\).
Heisenberg’s uncertainty principle requires that each TF atom must have: \( \Delta t \cdot \Delta x \geq 2\pi \).

Thus, for an optimal localization, the “life time” of an atom must influence its scale or frequency content.
In *Multiresolution Analysis* (MRA) a *scaling function* is used to create a series of approximations of a function (image), each differing by a factor of two.

Additional functions, *wavelets*, are used to encode the difference between adjacent approximations.
Wavelets and Multiresolution Processing

Series Expansions

- Express a signal $f(x)$ as

$$f(x) = \sum_{k} \alpha_k \varphi_k(x)$$

  expansion functions

  expansion coefficients

- If the expansion is unique, the $\varphi_k(x)$ are called **basis functions**, and the expansion set $\{\varphi_k(x)\}$ is called **basis**
Series Expansions

- All the functions expressible with this basis form a function space which is referred to as the closed span of the expansion set
  \[ V_{j_0} = \text{Span} \{ \varphi_k(x) \} \]

- If \( f(x) \in V \), then \( f(x) \) is in the closed span of \( \{ \varphi_k(x) \} \) and can be expressed as
  \[ f(x) = \sum_{k} \alpha_k \varphi_k(x) \]
For any expansion set \( \{\varphi_k(x)\} \) there exists a set of dual functions \( \{\tilde{\varphi}_k(x)\} \) that can be used to compute the coefficients \( \alpha_k \) as

\[
\alpha_k = \langle \tilde{\varphi}_k(x), f(x) \rangle = \int \tilde{\varphi}_k^*(x) f(x) \, dx
\]

Depending on the orthogonality of the expansion set, the computation assumes one of three possible forms.
Orthonormal Basis

- The expansion functions form an orthonormal basis for \( V \)

\[
\langle \varphi_j(x), \varphi_k(x) \rangle = \delta_{jk} = \begin{cases} 
0 & j \neq k \\
1 & j = k 
\end{cases}
\]

- The basis and its dual are equivalent, i.e.,

\( \varphi_k(x) = \tilde{\varphi}_k(x) \) and

\[
\alpha_k = \langle \varphi_k(x), f(x) \rangle = \int \varphi_k^*(x) f(x) \, dx
\]
Wavelets and Multiresolution Processing

Orthonormal Basis

- The expansion functions form an orthogonal basis for $V$
  \[ \langle \varphi_j(x), \varphi_k(x) \rangle = 0 \quad j \neq k \]
- The basis functions and their duals are called biorthogonal; they satisfy
  \[ \langle \varphi_j(x), \tilde{\varphi}_k(x) \rangle = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} \]
- The coefficients $\alpha_k$ are computed as
  \[ \alpha_k = \langle \tilde{\varphi}_k(x), f(x) \rangle = \int \tilde{\varphi}_k^*(x) f(x) \, dx \]
If the expansion set is not a basis for $V$, but supports the expansion

$$f(x) = \sum_k \alpha_k \varphi_k(x)$$

there is more than one set of $\alpha$ for any $f(x) \in V$. 

The expansion functions and their duals are said to be overcomplete or redundant; they form a frame in which

$$A\|f(x)\|^2 \leq \sum_k |\langle \varphi_k(x), f(x) \rangle|^2 \leq B\|f(x)\|^2$$

for some $A>0$ and $B<\infty$ and all $f(x) \in V$. 

Consider the set of expansion functions composed of integer translations and binary scalings of the real square-integrable function $\varphi(x)$ defined by

$$\{ \varphi_{j,k}(x) \} = \left\{ 2^{j/2} \varphi(2^j x - k) \right\}$$

for all $j, k \in \mathbb{Z}$ and $\varphi(x) \in L^2(\mathbb{R})$.

By choosing the scaling function $\varphi(x)$ wisely, $\{ \varphi_{j,k}(x) \}$ can be made to span $L^2(\mathbb{R})$. 
Scaling Function

\[ \varphi_{j,k}(x) = 2^{j/2} \varphi\left(2^j x - k\right) \]

- Index \( k \) determines the position of \( \varphi_{j,k}(x) \) along the \( x \)-axis, index \( j \) determines its width; \( 2^{j/2} \) controls its height or amplitude
- By restricting \( j \) to a specific value \( j = j_0 \) the resulting expansion set \( \{\varphi_{j_0,k}(x)\} \) is a subset of \( \{\varphi_{j,k}(x)\} \); as such, it does not span \( L^2(\mathbb{R}) \), but a subset within it
- One can write
  \[ V_{j_0} = \text{Span}\{\varphi_{j_0,k}(x)\}_{k} \]
- if \( f(x) \in V_{j_0} \) then \( f(x) = \sum_{k} \alpha_k \varphi_{j_0,k}(x) \)
The Haar Scaling Function

\[ \phi_{0,0}(x) = \phi(x) \]

\[ \phi_{0,1}(x) = \phi(x - 1) \]

\[ \phi_{1,0}(x) = \sqrt{2} \phi(2x) \]

\[ \phi_{1,1}(x) = \sqrt{2} \phi(2x - 1) \]

\[ f(x) \in V_1 \]

\[ \phi_{0,0}(x) \in V_1 \]

Any \( V_0 \) expansion function can be represented by \( V_1 \) expansion functions, i.e. \( V_0 \subset V_1 \).

\[ \phi_{0,k}(x) = \frac{1}{\sqrt{2}} \phi_{1,2k}(x) + \frac{1}{\sqrt{2}} \phi_{1,2k+1}(x) \]

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Basis for Vector space $V^j$

- Basis functions for $V^j$ are called **scaling functions** and are denoted by $\phi$.
- A simple basis for $V^j$ is given by the set of scaled and translated **box functions**:
  \[
  \phi^j(x) := \phi \left( 2^j x - i \right) \quad i = 0, \ldots, 2^j - 1
  \]
  where
  \[
  \phi(x) := \begin{cases} 
  1, & \text{for } 0 \leq x < 1 \\
  0, & \text{otherwise}
  \end{cases}
  \]

- Example basis for $V^2$:
MRA requirements 1: The scaling function is orthogonal to its integer translates.

MRA requirement 2: The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales \( V_{-\infty} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{\infty} \).

MRA requirement 3: The only functions that is common to all \( V_j \) is \( f(x) = 0 \) \( V_{-\infty} = \{0\} \).

MRA requirement 4: Any function can be represented with arbitrary precision \( V_{\infty} = \{L^2(\mathbb{R})\} \).
Under the previous conditions, the expansion functions of subspace $V_j$ can be expressed as a weighted sum of the expansion functions of subspace $V_{j+1}$:

$$\varphi_{j,k}(x) = \sum_n \alpha_n \varphi_{j+1,n}(x)$$

or

$$\varphi_{j,k}(x) = \sum_n h_\varphi(n) 2^{(j+1)/2} \varphi(2^{j+1} x - n)$$
Wavelets and Multiresolution Processing

Refinement (or MRA or Dilation) Equation

- Since \( \varphi(x) = \varphi_{0,0}(x) \), we can set both \( j \) and \( k \) to 0 and write

  \[
  \varphi(x) = \sum_{n} h_{\varphi}(n) \sqrt{2} \varphi(2x - n)
  \]

  scaling function coefficients

- The expansion functions of any subspace can be built from double-resolution copies of themselves, i.e., from expansion functions of the next higher resolution space.
For the Haar function

\[ \varphi(x) = \begin{cases} 
1 & 0 \leq x < 1 \\
0 & \text{otherwise}
\end{cases} \]

the scaling function coefficients are

\[ h_\varphi(0) = h_\varphi(1) = \frac{1}{\sqrt{2}} \quad H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \]

and it is

\[ \varphi(x) = \frac{1}{\sqrt{2}} \left[ \sqrt{2} \varphi(2x) \right] + \frac{1}{\sqrt{2}} \left[ \sqrt{2} \varphi(2x - 1) \right] \]
Given a scaling function which satisfies the four MRA requirements, one can define a wavelet function $\psi(x)$ which, together with its integer translates and binary scalings, spans the difference between any two adjacent scaling subspaces $V_j$ and $V_{j+1}$.

\[ V_{j+1} = V_j \oplus W_j \]

$V_j$ is the orthogonal complement of $V_j$ in $V_{j+1}$.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelet Functions

- Define the wavelet set
  \[
  \{\psi_{j,k}(x)\} = \left\{2^{j/2}\psi(2^j x - k)\right\}
  \]
  for all \(k \in \mathbb{Z}\) that spans the \(V_j\) spaces

- We write
  \[
  V_j = \text{Span}\left\{\psi_{j,k}(x)\right\}_k
  \]
  and, if \(f(x) \in W_j\)

  \[
  f(x) = \sum_k \alpha_k \psi_{j,k}(x)
  \]
Orthogonality

- This implies that
  \[ \left\langle \varphi_{j,k}(x), \psi_{j,l}(x) \right\rangle = 0 \]
  for all appropriate \( j, k, l \in \mathbb{Z} \)

- We can write
  \[ L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \]
  and also
  \[ L^2(\mathbb{R}) = \cdots \oplus W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \cdots \]
  (no need for scaling functions, only wavelets!)
Like scaling functions, wavelet functions can be expressed as a weighted sum of shifted double-resolution scaling functions:

\[ \psi(x) = \sum_n h_{\psi}(n) \sqrt{2} \varphi(2x - n) \]

with

\[ h_{\psi}(n) = (-1)^n h_{\varphi}(1 - n) \]
The scaling function coefficients are

\[ h_\varphi(0) = h_\varphi(1) = \frac{1}{\sqrt{2}} \]

By using \( h_\psi(n) = (-1)^n h_\varphi(1 - n) \), it is

\[ h_\psi(0) = (-1)^0 h_\varphi(1 - 0) = \frac{1}{\sqrt{2}} \]
\[ h_\psi(0) = (-1)^1 h_\varphi(1 - 1) = -\frac{1}{\sqrt{2}} \]

Hence

\[ \psi(x) = \varphi(2x) - \varphi(2x - 1) \]
Example: Haar Wavelet Functions in $W_0$ and $W_1$

$$\psi(x) = \psi_{0,0}(x)$$

$$\psi_{0,2}(x) = \psi(x - 2)$$

$$\psi_{1,0}(x) = \sqrt{2}\psi(2x)$$

$$f(x) \in V_1 = V_0 \oplus W_0$$

$$f_d(x) \in V_0$$

$$f_d(x) \in W_0$$

$$f(x) = f_a(x) + f_d(x)$$

(low frequencies) $

(high frequencies$)

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelet Series Expansions

- A function $f(x) \in L^2(\mathbb{R})$ can be expressed as

$$f(x) = \sum_{k} c_{j_0}(k)\varphi_{j_0,k}(x) + \sum_{j=j_0}^\infty \sum_{k} d_j(k)\psi_{j,k}(x)$$

approximation or scaling coefficients

$$c_{j_0}(k) = \left\langle f(x), \varphi_{j_0,k}(x) \right\rangle$$

detail or wavelet coefficients

$$d_j(k) = \left\langle f(x), \psi_{j,k}(x) \right\rangle$$
Example: The Haar Wavelet Series Expansion of \( y = x^2 \)

- Consider \( y = \begin{cases} 
  x^2 & 0 \leq x < 1 \\
  0 & \text{otherwise} 
\end{cases} \)

- The expansion coefficients are

\[
\begin{align*}
  c_0(0) &= \int_0^1 x^2 \varphi_{0,0}(x) \, dx = \frac{1}{3} \\
  d_0(0) &= \int_0^1 x^2 \psi_{0,0}(x) \, dx = -\frac{1}{4} \\
  d_1(0) &= \int_0^1 x^2 \psi_{1,0}(x) \, dx = -\frac{\sqrt{2}}{32} \\
  d_1(1) &= \int_0^1 x^2 \psi_{1,1}(x) \, dx = -\frac{3\sqrt{2}}{32}
\end{align*}
\]

\[
y = \frac{1}{3} \varphi_{0,0}(x) + \left[ -\frac{1}{4} \psi_{0,0}(x) \right] + \left[ -\frac{\sqrt{2}}{32} \psi_{1,0}(x) - \frac{3\sqrt{2}}{32} \psi_{1,1}(x) \right] + \cdots
\]

\[
V_1 = V_0 \oplus W_0 \\
V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1
\]
Example: The Haar Wavelet Series Expansion of $y=x^2$

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
• Let \( f(x), x = 0, 1, \ldots, M - 1 \) denote a discrete function.

• Its DWT is defined as

\[
\phi(j_0, k) = \frac{1}{\sqrt{M}} \sum_x f(x) \phi_{j_0, k}(x)
\]

\[
\psi(j, k) = \frac{1}{\sqrt{M}} \sum_x f(x) \psi_{j, k}(x) \quad j \geq j_0
\]

\[
f(x) = \frac{1}{\sqrt{M}} \sum_k W_\phi(j_0, k) \phi_{j_0, k}(x) + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi(j, k) \psi_{j, k}(x)
\]
Example: computing the DWT

- Consider the discrete function
  \[ f(0) = 1, f(1) = 4, f(2) = -3, f(3) = 0 \]

- It is \( M = 4 = 2^2 \rightarrow J = 2 \)

- The summations are performed over
  
  \[ x = 0, 1, 2, 3, j = 0, 1 \text{ and } k = 0 \text{ for } j = 0 \text{ or } k = 0, 1 \text{ for } j = 1 \]

- Use the Haar scaling and wavelet functions
Example: computing the DWT

\[ \varphi(0,0) = \frac{1}{2} \sum_{x=0}^{3} f(x) \varphi_{0,0}(x) = \frac{1}{2} \left[ 1 \cdot 1 + 4 \cdot 1 - 3 \cdot 1 + 0 \cdot 1 \right] = 1 \]

\[ \psi(0,0) = \frac{1}{2} \left[ 1 \cdot 1 + 4 \cdot 1 - 3 \cdot (-1) + 0 \cdot (-1) \right] = 4 \]

\[ \psi(1,0) = \frac{1}{2} \left[ 1 \cdot \sqrt{2} + 4 \cdot (-\sqrt{2}) - 3 \cdot 0 + 0 \cdot 0 \right] = -1.5 \sqrt{2} \]

\[ \psi(1,1) = \frac{1}{2} \left[ 1 \cdot 0 + 4 \cdot 0 - 3 \cdot \sqrt{2} + 0 \cdot (-\sqrt{2}) \right] = -1.5 \sqrt{2} \]
Example: computing the DWT

- The DWT of the 4-sample function relative to the Haar wavelet and scaling function thus is \[ \left\{ 1, 4, -1.5\sqrt{2}, -1.5\sqrt{2} \right\} \]

- The original function can be reconstructed as

\[
f(x) = \frac{1}{2} \left[ W_{\varphi}(0, 0)\varphi_{0,0}(x) + W_{\psi}(0, 0)\psi_{0,0}(x) + W_{\psi}(1, 0)\psi_{1,0}(x) + W_{\psi}(1, 1)\psi_{1,1}(x) \right]
\]

for \( x = 0, 1, 2, 3 \)
Consider the multiresolution refinement equation
\[ \varphi(x) = \sum_{n} h_{\varphi}(n) \sqrt{2} \varphi(2x - n) \]

Scaling \( x \) by \( 2^j \), translating it by \( k \), and letting \( m = 2k + n \) gives
\[
\varphi(2^j x - k) = \sum_{n} h_{\varphi}(n) \sqrt{2} \varphi(2(2^j x - k) - n)
\]
\[= \sum_{m} h_{\varphi}(m - 2k) \sqrt{2} \varphi(2^{j+1} x - m) \]
Deriving the FWT

- Similarly, it is

\[ \psi(2^j x - k) = \sum_m h_\psi(m - 2k)\sqrt{2}\varphi(2^{j+1} x - m) \]

whence

\[ \psi(j, k) = \frac{1}{\sqrt{M}} \sum_x f(x)\psi_{j,k}(x) \]

\[ = \frac{1}{\sqrt{M}} \sum_x f(x)2^{j/2}\psi(2^j x - k) \]

\[ = \frac{1}{\sqrt{M}} \sum_x f(x)2^{j/2} \left[ \sum_m h_\psi(m - 2k)\sqrt{2}\varphi(2^{j+1} x - m) \right] \]
Deriving the FWT

- This can be rewritten as
  \[ \psi(j, k) = \sum_m h_\psi(m - 2k) \left[ \frac{1}{\sqrt{M}} \sum_x f(x) 2^{(j+1)/2} \phi(2^{j+1} x - m) \right] \]
  \[ = \sum_m h_\psi(m - 2k) W_\phi(j + 1, m) \]
- And similarly, it can be shown that
  \[ \phi(j, k) = \sum_m h_\phi(m - 2k) W_\phi(j + 1, m) \]
- These equations are equivalent to convolutions with kernels \( h_\psi(-n) \) and \( h_\phi(-n) \) followed by subsampling by a factor of 2
Thus we can write

\[ \psi(j, k) = h_\psi(-n) \ast W_\phi(j + 1, n) \bigg|_{n=2k, k \geq 0} \]

\[ \phi(j, k) = h_\phi(-n) \ast W_\phi(j + 1, n) \bigg|_{n=2k, k \geq 0} \]

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Two-Stage (or Two-Scale) FWT Analysis Bank

\[ f(n) = W_\varphi(J, n) \]

\[ h_\psi(-n) \quad 2\downarrow \quad W_\psi(J-1, n) \]

\[ h_\varphi(-n) \quad 2\downarrow \quad W_\varphi(J-1, n) \]

\[ h_\varphi(-n) \quad 2\downarrow \quad W_\varphi(J-2, n) \]

\[ |H(\omega)| \]

\[ 0 \quad \pi/4 \quad \pi/2 \quad \pi \]

\[ V_{J-1} \quad V_J \quad V_{J-2} \quad W_{J-2} \quad W_{J-1} \]

FIGURE 7.16
(a) A two-stage or two-scale FWT analysis bank and (b) its frequency splitting characteristics.
Consider the function $f(n) = \{1, 4, -3, 0\}$.
$\varphi(j + 1, k) = h_\varphi(k) * W_{\varphi}^{up}(j, k) + h_\psi(k) * W_{\psi}^{up}(j, k) \bigg|_{k \geq 0}$
Two-Stage (or Two-Scale) FWT\(^{-1}\) Synthesis Bank

**FIGURE 7.19** A two-stage or two-scale FWT\(^{-1}\) synthesis bank.
**Example: Computing a 1-D FWT**

**FIGURE 7.20** Computing a two-scale inverse fast wavelet transform of sequence \(\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\}\) with Haar scaling and wavelet vectors.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
**Wavelets and Multiresolution Processing**

**FWT vs. FFT**

- **Computational complexity:**
  - FWT of a length $M = 2^J$ sequence: $O(M)$
  - FFT of a length $M = 2^J$ sequence: $O(M \log M)$

- **Heisenberg Uncertainty Principle for Information Theory:**

![Time-frequency tilings for sampled data, FFT, and FWT basis functions.](Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
2-D DWT: Definition

- Define the scaled and translated basis functions

\[
\varphi_{j,m,n}(x,y) = 2^{j/2} \varphi(2^j x - m, 2^j y - n)
\]

\[
\psi^i_{j,m,n}(x,y) = 2^{j/2} \psi^i(2^j x - m, 2^j y - n), \quad i = \{H, V, D\}
\]

- Then

\[
\varphi(j_0, m, n) = \frac{1}{\sqrt{M N}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \varphi_{j_0, m, n}(x,y)
\]

\[
\psi^i(j, m, n) = \frac{1}{\sqrt{M N}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \psi^i_{j, m, n}(x,y), \quad i = \{H, V, D\}
\]

\[
f(x,y) = \frac{1}{\sqrt{M N}} \sum_{m} \sum_{n} W_{\varphi}(j_0, m, n) \varphi_{j_0, m, n}(x,y)
\]

\[
+ \frac{1}{\sqrt{M N}} \sum_{i=H,V,D} \sum_{j=j_0}^{\infty} \sum_{m} \sum_{n} W_{\psi}^i(j, m, n) \psi^i_{j, m, n}(x,y)
\]
Wavelets and Multiresolution Processing

2-D DWT: Analysis and Synthesis Filter Banks

Analysis Filter Banks

Resulting Decomposition

Synthesis Filter Banks

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Example: A Three - Scale FWT

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelets and Multiresolution Processing

Analysis and Synthesis Filters

**Figure 7.24**
Fourth-order symlets:
(a)–(b) decomposition filters;
(c)–(d) reconstruction filters;
(e) the one-dimensional wavelet; (f) the one-dimensional scaling function;
and (g) one of three two-dimensional wavelets, $\psi^H(x, y)$.

Wavelet function

Analysis filters

Synthesis filters

Scaling function

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelets and Multiresolution Processing

\[ \psi^H (x, y) = \psi(x) \phi(y) \]
Example: Wavelet-Based Edge Detection

FIGURE 7.25
Modifying a DWT for edge detection: (a) and (c) two-scale decompositions with selected coefficients deleted; (b) and (d) the corresponding reconstructions.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Example: Modifying a DWT for Noise Removal

Figure 7.26
Modifying a DWT for noise removal: (a) a noisy MRI of a human head; (b), (c), and (e) various reconstructions after thresholding the detail coefficients; (d) and (f) the information removed during the reconstruction of (c) and (e). (Original image courtesy Vanderbilt University Medical Center.)
Haar Wavelet Function Coefficients

Haar scaling filter: $h_\varphi(0) = h_\varphi(1) = 1/\sqrt{2}$

Haar wavelet filter: $h_\psi(0) = 1/\sqrt{2}; h_\psi(1) = -1/\sqrt{2}$

Haar wavelet function: $\psi(x) = \begin{cases} 1 & 0 \leq x < 0.5 \\ -1 & 0.5 \leq x < 1 \\ 0 & \text{elsewhere} \end{cases}$

$f(x) = f_a(x) + f_d(x)$

$$f_a(x) = \frac{3\sqrt{2}}{4} \varphi_{0,0}(x) - \frac{\sqrt{2}}{8} \varphi_{0,2}(x)$$

$$f_d(x) = -\frac{\sqrt{2}}{4} \psi_{0,0}(x) - \frac{\sqrt{2}}{8} \psi_{0,2}(x)$$

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelet Series Expansions

Consider the orthogonal case

\[ f(x) = \sum_k c_{j_0}(k) \phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k) \psi_{j,k}(x) \]

\[ c_{j_0}(k) = \langle f(x), \phi_{j_0,k}(x) \rangle \]

\[ d_j(k) = \langle f(x), \psi_{j,k}(x) \rangle \]

\(c(k)\) are referred as approximation coefficients and \(d(k)\) are referred as detail coefficients.

Example: \( f(x) = \begin{cases} x^2 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \)

**Figure 7.13** A wavelet series expansion of \( y = x^2 \) using Haar wavelets.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelet Packets

- The Wavelet Transform decomposes a function into a series of logarithmically related frequency bands. Low frequencies are grouped into narrow bands, while the high frequencies are grouped into wider bands.

- *Wavelet Packets* are a generalization that allows greater control over the time-frequency plane partitioning.

- Consider the two-scale filter bank as a *binary tree*. The Wavelet coefficients are at the nodes of the tree. The *root node* contains the highest-scale approximation coefficients (i.e., the sampled signal itself). Each node contains coefficients representing different subspaces — *subspace analysis tree*.

![Wavelet Packets Diagram](Image)

**FIGURE 7.27** A coefficient (a) and analysis (b) tree for the two-scale FWT analysis bank of Fig. 7.16.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelets and Multiresolution Processing

WT: Filter Bank, Analysis Tree, and Spectrum Splitting

FIGURE 7.28 A three-scale FWT filter bank: (a) block diagram; (b) decomposition space tree; and (c) spectrum splitting characteristics.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelet packets are conventional wavelet transforms in which the *details* are also iteratively filtered.

Subscripts in the figure show the scale and a string of *A*’s and *D*’s encoding the path from the parent to the node.

The packet tree almost triples the number of decompositions.

**FIGURE 7.29** A three-scale wavelet packet analysis tree.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelets and Multiresolution Processing

Wavelet Packets: Filter Structure and Spectrum Splitting

\[ f(x) \in V_J \]

\[ H[\omega] \]

\[ 0 \quad \pi/8 \quad \pi/4 \quad \pi/2 \quad \pi \]

Fig. 7.30 The (a) filter bank and (b) spectrum splitting characteristics of a three-scale full wavelet packet analysis tree.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Wavelets and Multiresolution Processing

Wavelet Packets

FIGURE 7.31 The spectrum of the decomposition in Eq. (7.6-5).

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
In 2-D, separable implementation of wavelet packets leads to a quad-tree structure.

The frequency spectrum is divided into four areas. The low-frequency area is in the centre.

**FIGURE 7.32** The first decomposition of a two-dimensional FWT: (a) the spectrum and (b) the subspace analysis tree.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Portion of 2-D Wavelet Packet Tree

**FIGURE 7.33** A three-scale, full wavelet packet decomposition tree. Only a portion of the tree is provided.
Wavelet Packet Decomposition

**FIGURE 7.34** (a) A scanned fingerprint and (b) its three-scale, full wavelet packet decomposition. (Original image courtesy of the National Institute of Standards and Technology.)

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
**Entropy-based** criteria for searching the optimal (the best) wavelet packet decomposition

An *additive cost function* for comparing different decompositions

**FIGURE 7.36** The optimal wavelet packet analysis tree for the decomposition in Fig. 7.35.

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)
Optimal Wavelet Packet Basis

FIGURE 7.35 An optimal wavelet packet decomposition for the fingerprint of Fig. 7.34(a).

(Images from Rafael C. Gonzalez and Richard E. Wood, Digital Image Processing, 2nd Edition.)